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Research Article

Stability of FGP beams under thermo-electro-mechanical loading

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A b s t r a c t

In this paper the buckling of Functionally Graded Piezoelectric (FGP) beam has been investigated under thermal, electrical and mechanical loadings. The beam is assumed to be graded through the thickness by the simple power law. The governing equations has been obtained using principle of minimum potential energy based on Euler-Bernoulli beam assumptions. The buckling analysis of FGP beam has been performed for various types of boundary conditions and the effect of the electrical and thermal loadings on the stability of the beam has been investigated. The results show that thermal loading has more effect on the buckling point of the beam in compares to the electrical loading. Also it is seen that the buckling load increases by increasing the power law index for two cases of thermal and electrical loadings.

Keywords:

Stability Analysis,
Functionally graded Piezoelectric Material,
Euler-Bernoulli beam theory,
Electrical and Thermal Loadings.

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1. Introduction

Nowadays, piezoelectric materials have many applications especially as sensors and actuators in the structures. The coupling of mechanical and electrical properties is the main advantage of such materials. Piezoelectric materials are widely used in vibration control, buckling prediction and health monitoring of structures. Additionally, using of these materials in thermal environments takes more attention in research of structures. This is because of the stress concentration and free edge stress phenomenon that is occurred in the surfaces between piezoelectric layers in structures. Creating the idea of combining the two types of smart materials defines the functionally graded piezoelectric materials (FGPM). There are many works that have been done in the field of functionally graded materials, and in this section some of them are mentioned.

Li & Song [1] studied the large thermal deflection of Timoshenko beams under transversely non-uniform temperature rise. The equilibrium equations were derived with considering geometrically non-linear deformation. Shooting method was used to solve the problem. Song & Li [2] studied thermal buckling and post-buckling of Euler-Bernoulli beam on elastic foundation. Linear and non-linear behavior of the beams were taken into account. The geometric method is used to derive the equations. They solved the problems using shooting method.

Li, et al [3] presented the post-buckling of functionally graded Timoshenko beams. Power law distribution in thickness direction was considered to express of material property. Also, the geometric method was used to drive equilibrium equation. The beam made of inhomogeneous-isotropic materials that caused the change of shear modulus in the thickness direction. They using shooting method to search for numerical solution. Kiani & Eslami [4] investigated on thermal buckling of functionally graded Euler-Bernoulli beam. They assumed power law distribution for material property in thickness direction. The adjacent-equilibrium criterion was used for linearization of governing equation. Finally, exact solution of stability problem was obtained from the system of equations. In this work, the effect of three type of thermal loading were investigated on critical buckling temperature of FGM beams. Wattanasakulpong, et al [5] investigated the thermal buckling of the FGM beams using third-order shear deformation beam theory. The material properties were assumed to vary smoothly and continuously across the thickness of beam and the Ritz method was adopted to solve the eigenvalue problem that associated with thermal buckling in various types of immovable boundary conditions. Rahimi, et al., [6] studied the free vibration and postbuckling behavior of functionally graded beam using Timoshenko beam theory. They investigated the postbuckling deformation as a function of the exerted axial load by means of an exact solution method.

In additional, there are many works done in the field of analysis of piezoelectric structures. Such as, Shi & Chen [7] investigated the functionally graded piezoelectric cantilever beam under different loadings. They used a stress function to solve the governing equations of the beam. Yang & Xiang [8] studied the static bending, free vibration and dynamic response of actuators made of FGPM under combined thermo-electro-mechanical loading. They used Timoshenko beam theory. Doroushi, et al., [9] developed the finite element method to investigate the response of free and forced vibration characteristics of a FGP beam under thermo-electro-mechanical loading. They used the higher-order shear deformation beam theory. Komeili, et al., [10] presented the analysis of static bending of beams made of FGPMs. The results was shown for three type of beam's theories (Euler-Bernoulli beam theory (EBT), first-order shear deformation theory (FSDT) and third-order shear deformation theory (TSDT)) in thermo-electro-mechanical loading. The finite element method and Fourier series were used to solve the governing equations. Kiani, et al [11] investigated thermo-electrical buckling of piezoelectric functionally graded beam based on Timoshenko beam theory. Defining the material properties were expressed by power law distribution along the thickness direction. The electric field in piezoelectric layer was assumed constant. The result obtained for three types of thermal loading. Komijani, et al [12] studied non-linear thermo-electrical stability of FGPM beam using Timoshenko beam theory. All of thermo-electro-mechanical properties were assumed to vary in thickness direction and expressed by power law distribution. The Ritz finite element method was used to solve the governing equation. Bodaghi, et al., [13] investigated the geometrically non-linear static and dynamic analysis of functionally graded beams with a pair of functionally graded piezoelectric layers as sensor using Timoshenko beam theory. The solution is obtained using the hybrid generalized differential quadrature method, Newmark algorithm, Newton-Raphson iterative scheme. Li, et al., [14] presented bending and free vibration analysis of functionally graded piezoelectric beam. The beam is based on modified strain gradient theory. Defining of material properties were based on power law distribution along the thickness direction. The Fourier series method was used to solve the governing equation.

In this article, the thermo-electro-mechanical buckling of functionally graded piezoelectric beam is discussed by using Euler-Bernoulli beam theory. Defining of material properties are derived base on power law model in thickness direction. The beam is subjected to axial force and electrical force with constant voltage along thickness direction and two types of thermal loading. Five types of boundary conditions are considered for the beam. Governing equation are obtained for this beam by using the minimum potential energy principle. By solving trivial and non-trivial solution, stability and instability of FGPM beam is discussed. Also, bifurcation point are shown in tables for various of power law index (k) and boundary conditions.

2. Functionally graded piezoelectric beam

Consider a functionally graded piezoelectric beam with rectangular cross section, length L, width b and thickness h. This beam is subjected to axial force P in the x direction, distributed load q and constant voltage V_0 through thickness shown in Fig.1.

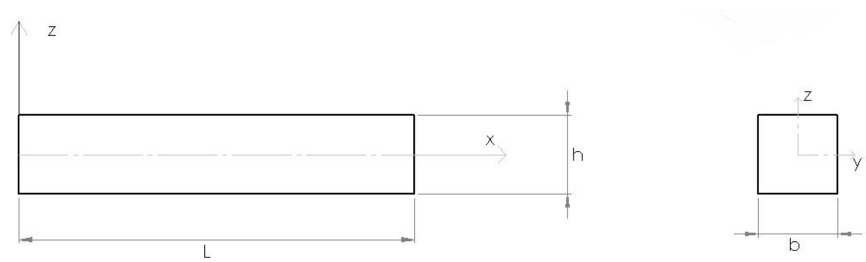

Fig.1: Coordinate system and geometry of FGPM beam

Assuming a continuous distribution of material properties along the thickness that can be obtained from power law distribution equation:

$$P(z) = P_L + P_{UL}\left(\frac{1}{2}+\frac{z}{h}\right)^k \qquad (1)$$

Here, $P_{UL} = P_U - P_L$, P_L and P_U are material properties in bottom surface and top surface respectively. In this article, material properties such as Young's modulus *E(z)*, coefficient of thermal expansion *α(z)* and piezoelectric coefficient *e(z)* are defined varying in thickness direction according to equation (1). With a constant voltage applied to the thickness direction, the electric field can be obtained as [11],

$$E_z = -\frac{V_0}{h} \qquad (2)$$

3. Governing equation

Consider a FGP beam as shown in Fig.1. The rectangular Cartesian coordinate system is used such that the x-axis defined the length of beam on the middle surface and the thickness direction is z-axis (Fig.1). The formulation is based on using Euler-Bernoulli beam theory. The displacement field can be expressed by [4]

$$\begin{aligned} \hat{u}(x,z) &= u(x) - zw_{,x} \\ \hat{w}(x,z) &= w(x) \end{aligned} \qquad (3)$$

Where $u(x), w(x)$ are displacement of an arbitrary point of the beam along the length and thickness directions, respectively. Von-Karman type non-linear strain-displacement relations can be obtained as bellow

$$S_x = \hat{u}_{,x} + \frac{1}{2}\left(\hat{w}_{,x}\right)^2 \qquad (4)$$

Where S_x is axial strain. Substituting Eq (3) into Eq (4) gives

$$S_x = u_{,x} + \frac{1}{2}\left(w_{,x}\right)^2 - zw_{,xx} \qquad (5)$$

Furthermore the beam is subjected to thermal loading. The constitutive equations for FGP beam are given by [12]

$$\begin{aligned} \sigma_x &= Q_{11}(z)\left(S_x - \alpha(z)\Delta T\right) - e_{13}(z)E_z \\ D_z &= e_{31}(z)S_x + \varepsilon_{33}(z)E_z + p_3\Delta T \end{aligned} \qquad (6)$$

Where D_z ,S_x ,σ_x and E_z represent electric displacement, axial strain, axial stress and corresponding electric field components, respectively. Here, e_{ij} , α_{ij} ,Q_{ij} , p_3 and ε_{ij} are the piezoelectric coefficient, thermal expansion coefficient, the elastic stiffness coefficient , the pyroelectric coefficient and dielectric coefficient.

The force and moment resultant per unit length of the FGP beam expressed in terms of the stress through the thickness, and according to Euler-Bernoulli beam theory, they are

$$\left(N_x, M_x\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \left(1, z\right) dz \tag{7}$$

With substituting Eq (6) into Eq (7) , the force and moment resultant per unit length of the beam obtained as:

$$\begin{aligned} N_x &= A_{11}\left(u_{,x} + \frac{1}{2}\left(w_{,x}\right)^2\right) - B_{11} w_{,xx} - N_x^T - N_x^e \\ M_x &= B_{11}\left(u_{,x} + \frac{1}{2}\left(w_{,x}\right)^2\right) - D_{11} w_{,xx} - M_x^T - M_x^e \end{aligned} \tag{8}$$

Where the extensional, coupling, and bending stiffness constants and the thermal and electrical force and moment resultants are defined as:

$$\left(A_{11}, B_{11}, D_{11}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}\left(z\right)\left(1, z, z^2\right) dz \tag{9}$$

$$\left(N_x^T, M_x^T\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}\left(z\right)\alpha\left(z\right)\Delta T \left(1, z\right) dz \tag{10}$$

$$\left(N_x^e, M_x^e\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} e_{13}\left(z\right) E_z \left(1, z\right) dz \tag{11}$$

In this study, the minimum potential energy principle is used to obtain the differential equations of equilibrium. According to this principle, the system is in equilibrium if

$$\delta\pi = \delta H + \delta W \tag{12}$$

Where H is electric enthalpy and W is virtual work of external forces acting on the beam. Variation of the electric enthalpy can be obtained as [12]

$$\delta H = \iiint_V [\sigma_x \delta S_x - D_z \delta E_z] dV \tag{13}$$

It should be noted that the electrical field variation is zero because the electrical field is constant (Eq (2)). The virtual work for axial force P and distributed load q are defined by

$$\delta W = -\int_0^L q\,\delta w dx - \bar{P}\delta u - \bar{M}\,\delta w_{,x} - \bar{R}\,\delta w \tag{14}$$

Where $\bar{P}, \bar{R}$ and $\bar{M}$ are resultant of axial and lateral force and moment at the two edge of beam and are defined as:

$$\begin{aligned} \bar{P} &= P - N_x^T - N_x^e \\ \bar{M} &= M - M_x^T - M_x^e \\ \bar{R} &= R - R_x^T - R_x^e \end{aligned} \tag{15}$$

Where $P, R$and M are resultant of axial and lateral external force and external moment at the two edges of the beam respectively.

Substituting Eq (13) and Eq (14) into Eq (12) and integrating respect to z and substituting in Eq (7), the equilibrium equations for ideal column are obtained

$$N_{x,x} = 0 \tag{16}$$

$$M_{x,xx} + \left(N_x w_{,x}\right)_{,x} = 0 \tag{17}$$

An ordinary differential equation in terms of w is obtained combining Eq (8) into Eq (16) and (17), which is the governing equation of FGPM beam as:

$$w_{,xxxx} + \mu^2 w_{,xx} = 0 \tag{18}$$

With

$$\mu^2 = \frac{A_{11}N_x}{B_{11}^2 - A_{11}D_{11}} \tag{19}$$

The corresponding boundary conditions are defined as bellow

$$N_x = \bar{P} \qquad or \qquad u(0) = u(L) \tag{20}$$

$$w_{,xxx} + \mu^2 w_{,x} = \bar{R} \qquad or \qquad w = 0 \tag{21}$$

$$M_x = \left(\frac{\left(B_{11}^2 - A_{11}D_{11}\right)w_{,xx} + B_{11}P}{A_{11}}\right) - M_x^T - M_x^e = 0 \qquad or \qquad w_{,x} = 0 \tag{22}$$

According to Eq (15) and Eq (20) μ^2 is equal to

$$\mu^2 = \frac{A_{11}\left(P - N_x^T - N_x^e\right)}{B_{11}^2 - A_{11}D_{11}} \tag{23}$$

The parameter μ is a function of thickness, so the exact solution of Eq (18)

$$w(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x) + C_3 x + C_4 \tag{24}$$

The constants of integration, C_1 to C_4 are obtained by using the boundary conditions of the beam. In this article, the boundary conditions are tabulated in Table 1. Since the aim of this study is to obtain the bifurcation point and stability of FGP beam, by solving Eq (18) the stability of FGP beam can be obtained.

Table *1*: boundary conditions for FGP Euler-Bernoulli beam

B.Cs.	at x=0
Clamped	$w = w_{,x} = 0$
Simply supported	$w = \left(B_{11}^2 - A_{11}D_{11}\right)w_{,xx} + B_{11}P - A_{11}(M_x^T + M_x^e) = 0$
Roller	$w_{,xxx} + \mu^2 w_{,x} = w_{,x} = 0$

The clamped-clamped boundary condition are defined as [15]

$$\begin{aligned} &u(0) = u(L) \quad or \quad N_x = \overline{P} \\ &w(0) = w(L) = 0 \\ &w_{,x}(0) = w_{,x}(L) = 0 \end{aligned} \tag{25}$$

Satisfying the boundary conditions is leaded to four linear algebraic equations in the four constants C_i.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin(\mu L) & \cos(\mu L) & L & 1 \\ \mu & 0 & 1 & 0 \\ \mu\cos(\mu L) & -\mu\sin(\mu L) & 1 & 0 \end{bmatrix} \times \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

Assuming four constants are not equal to zero, then lead to a non-trivial solution. To solve the problem the determinant of coefficient matrix set to be zero, which yields

$$2\left(\cos(\mu L) - 1\right) + \mu L \sin(\mu L) = 0 \tag{27}$$

By solving Eq (27), the bifurcation point is obtained as:

$$\left(N_x\right)_{cr} = \frac{B_{11}^2 - A_{11}D_{11}}{A_{11}} \times \frac{4n^2\pi^2}{L^2} \tag{28}$$

According to Eq (15) and Eq (20), the minimum buckling load is obtained

$$\left(P - N_x^T - N_x^e\right)_{cr} = \frac{B_{11}^2 - A_{11}D_{11}}{A_{11}} \times \frac{4\pi^2}{L^2} \tag{29}$$

According to Eq (29), the bifurcation point is affected of mechanical, electrical and thermal load.

In the second example, consider a FGP beam with both edges simply supported. Satisfying the simply supported-simply supported of boundary conditions is leaded to four linear algebraic equations in the four constants C_i (see Eq (30)).

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\mu^2 & 0 & 0 \\ \sin(\mu L) & \cos(\mu L) & L & 1 \\ -\mu^2 \sin(\mu L) & -\mu^2 \cos(\mu L) & 0 & 0 \end{bmatrix} \times \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \times \frac{A_{11}(M_x^T + M_x^e) - B_{11}P}{B_{11}^2 - A_{11}D_{11}} \tag{30}$$

The deflection of the beam is obtained by Eq (31) which has trivial solution

$$w(x) = \frac{B_{11}P - A_{11}(M_x^T + M_x^e)}{A_{11}N_x}\left[\left(\frac{1-\cos(\mu L)}{\sin(\mu L)}\right)\sin(\mu x) + \cos(\mu x) - 1\right] \tag{31}$$

It’s worth to note that, the functionally graded piezoelectric beam is a non-homogeneous materials, hence, the coupling stiffness matrix is not equal to zero. This reason lead to non-homogeneous boundary conditions. Hence, the linear algebraic equations from non-homogeneous boundary conditions have a trivial solution.

3.1. Loadings

In this article, the beam is subjected to three type of loadings: electrical, thermal and mechanical. The FGP beam is subjected two types of thermal loading as illustrated bellow:

In uniform loading, the beam is subjected to the uniform temperature rise $\Delta T = T - T_0$ that causes the beam buckle, where T_0 and T are initial temperature and final temperature. Substituting $T = T_0 + \Delta T$ into Eq (29), the bifurcation point is obtained. And in the second type, the beam subjected to linear temperature loading. Consider a thin FGPM beam which the temperature in lower surface and upper surface are T_L and T_U respectively. The temperature distribution for the given boundary conditions is defined by solving the heat conduction equation along the FGPM beam thickness. Assuming the thickness of beam is thin, the temperature distribution is approximated by linear function of thickness. So temperature distribution is become [11]

$$T = T_L + T_{UL}\left(\frac{1}{2} + \frac{z}{h}\right) \quad , \quad \Delta T = T - T_0 \qquad (32)$$

Substituting Eq (32) into Eq (10) and using Eq (29), the bifurcation point can be obtained.

4. Result and discussion

The FGP beam consists of PZT-4 and PZT-5H materials in upper and lower surface, respectively. The stability behavior of functionally graded piezoelectric beam is investigated in this section. Thermo-electro-mechanical properties of these materials are listed in Table 2.

Table 2: Thermo-electro-mechanical properties of PZT-4 and PZT-5H [12]

	PZT-4	PZT-5H
$Q_{11}(GPa)$	81.3	60.6
$e_{13}(C/m^2)$	-10.0	-16.604
$e_{15}(C/m^2)$	40.3248	44.9046
$\varepsilon_{11}(C^2/m^2N)\times10^{-8}$	0.6712	1.5027
$\varepsilon_{33}(C^2/m^2N)\times10^{-8}$	1.0275	2.554
$\alpha(1/K)$	2e-6	10e-6
$p_3\times10^{-5}$	2.5	0.548

In **Error! Reference source not found.**, the critical buckling resultant force $(P - N_x^T - N_x^e)_{cr}$ is plotted as a function of aspect ratio (L/h) of FGP beam with C-C boundary conditions for various power law index (*k*). It is seen that, as the aspect ratio decrease, the magnitude of buckling resultant force increase.

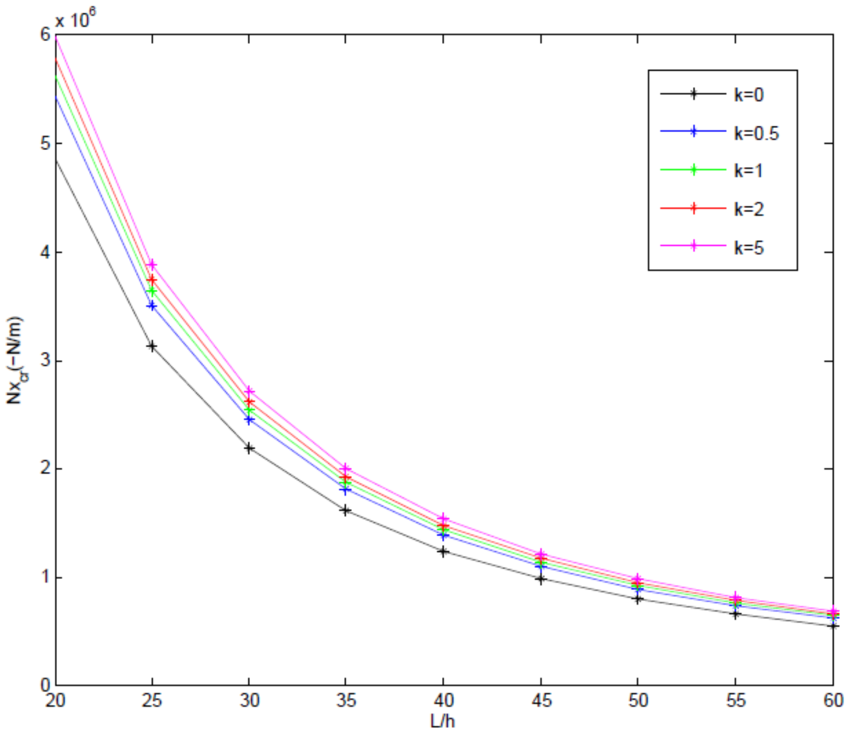

Fig. 2: The buckling resultant force versus aspect ratio of FGPM beam with C-C boundary conditions for various power law index

The critical buckling load for two types of temperature distribution is shown in Table 3. The constant voltage $(V = 500(v))$ is applied along the thickness. The Clamped-Clamped boundary condition is assumed for the beam (*L=0.25m, h=0.01m*). It is seen that, with increasing the power law index, the critical buckling load increases from the beam made of PZT-5H $(k = 0)$ to the beam made of PZT-4 $(k = \infty)$. Hence, as power law index increases, the critical buckling load increases. On the other hand, with decreasing the applied temperature, the magnitude of buckling load for specific k is increased.

Table *3*: Effect of temperature difference on buckling load (-MPa.m) C-C boundary condition (V=500 v, L=0.25m, h=0.01m)

	ΔT	k=0	k=0.5	k=1	k=2	k=5
Uniform temperature rise	200	1.969	2.597	2.877	3.147	3.452
	100	2.575	3.083	3.289	3.479	3.701
	0	3.181	3.569	3.701	3.812	3.950
	-100	3.787	4.055	4.113	4.144	4.199
	-200	4.393	4.541	4.525	4.477	4.448
	T_c					

Linear temperature rise(T_m=50)	200	2.545	3.008	3.198	3.387	3.631
	100	2.848	3.278	3.441	3.593	3.786
	0	3.151	3.547	3.684	3.799	3.940
	-100	3.454	3.817	3.927	4.005	4.095
	-200	3.757	4.087	4.170	4.211	4.249

In Table 4 ,Effect of applied voltage V_0 on critical buckling load for the uniform and linear case of temperature distribution through the thickness is shown. In this case, the C-C boundary condition is assumed for both end of FGP beam (*L=0.25 m*, *h=0.01 m*). It is seen that, as power law index increases in constant voltage, the critical buckling load increases. The critical buckling load for constant k is increased with decreasing the applied voltage. In the other hand, the magnitude of critical buckling load in uniform temperature rise is more than linear case. But in general, changes in the results are small.

Table *4*: Effect of applied voltage on buckling load (-MPa.m) of C-C FGP Beam (L=0.25m, h=0.01m)

	V_0	k=0	k=0.5	k=1	k=2	k=5
Uniform temperature rise(ΔT = 0)	500	3.1815	3.5695	3.7015	3.8122	3.9502
	200	3.1865	3.5738	3.7055	3.8159	3.9536
	0	3.1899	3.5767	3.7081	3.8183	3.9558
	-200	3.1932	3.5796	3.7108	3.8207	3.9580
	-500	3.1982	3.5839	3.7148	3.8244	3.9613
Linear temperature rise (T_m = 100 , T_c = 200)	500	2.5453	3.0085	3.1988	3.3877	3.6318
	200	2.5502	3.0129	3.2028	3.3914	3.6352
	0	2.5536	3.0157	3.2055	3.3938	3.6374
	-200	2.5569	3.0186	3.2081	3.3962	3.6396
	-500	2.5619	3.0229	3.2121	3.3999	3.6429

Effect of aspect ratio on critical buckling load for two types of temperature distribution is shown in Table 5. The results are tabulated for some power law index. It is seen that, as aspect ratio increases in constant power law index, the critical buckling load decreases. On the other hand, the value of critical buckling load for specific aspect ratio is increased with increasing the power law index. It is worth to note that, the magnitude of critical buckling load in the uniform case is more than linear case of temperature distribution.

Table 5: Effect of aspect ratio on buckling load (-MPa.m) of C-C FGP beam (V=500 v)

	L / h	**k=0**	**k=0.5**	**k=1**	**k=2**	**k=5**
Uniform temperature rise($\Delta T = 0$)	10	19.9282	22.3474	23.1693	23.8586	24.7183
	15	8.8524	9.9282	10.2938	10.6004	10.9828
	20	4.9758	5.5814	5.7873	5.9600	6.1754
	30	2.2068	2.4766	2.5684	2.6455	2.7415
	40	1.2377	1.3899	1.4418	1.4854	1.5396
	50	0.7891	0.8869	0.9203	0.9484	0.9834
	75	0.3461	0.3902	0.4053	0.4181	0.4339
Linear temperature rise ($T_m = 10$, $T_c = 100$)	10	19.5949	22.0561	22.9095	23.6400	24.5544
	15	8.5191	9.6368	10.0340	10.3818	10.8189
	20	4.6425	5.2901	5.5275	5.7414	6.0114
	30	1.8735	2.1853	2.3086	2.4269	2.5775
	40	0.9044	1.0986	1.1820	1.2668	1.3757
	50	0.4558	0.5956	0.6605	0.7298	0.8194
	75	0.0128	0.0989	0.1455	0.1995	0.2700

5. Conclusion

In this article, the buckling and stability of functionally graded piezoelectric beams are investigated under thermo-electro-mechanical loads. The beam is modelled based on Euler-Bernoulli beam theory. The existence of bifurcation point for five case of boundary conditions is studied. It is observed that the structure with homogeneous boundary conditions is unstable, and non-homogeneous boundary conditions cause to stable the FGP beam. The Clamped-Clamped and Clamped-Roller of boundary conditions are homogeneous. It is worth noting that the bifurcation point increase by increasing the power law index for any case of thermal and electrical loading. Effect of temperature difference, applied voltage and aspect ratio on buckling load are obtained for some power law index. It is seen that effect of applied voltage against the thermal force on bifurcation point is too low.

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